MTH 305: Practice assignment 3

1 Fundamental theorem of arithmetic

Establish the following assertions.

(i) The only prime of form form $n^3 - 1$ is 7.

(ii) The only prime p for which $3p + 1$ is a perfect square is $p = 5$.

(iii) The number $p^2 + 2$ is composite for ever prime $p \geq 5$.

(iv) If p is a prime and $p \mid a^n$, then $p^n \mid a^n$.

(v) If $n > 4$ is composite, then $n \mid (n-1)!$.

(vi) The number $8^n + 1$ is composite for every integer $n \geq 1$.

(vii) If p, q are primes such that $p \geq q \geq 5$, then $24 \mid p^2 - q^2$.

(viii) If $p \neq 5$ is an odd prime, then either $10 \mid p^2 - 1$ or $10 \mid p^2 + 1$.

(ix) If $n > 1$ is an integer not of the form $6k + 3$, then $n^2 + 2^n$ is composite.

(x) An integer is said to be *square-free* if is not divisible by the square of an integer greater than 1.

  (a) An integer $n > 1$ is square-free if and only if n can be factored to a product of distinct primes.

  (b) Every integer $n > 1$ is a product of a square-free integer and a perfect square.

2 Distribution of primes

Establish the following assertions.

(i) If $p \nmid n$ for all primes $p \leq \sqrt[3]{n}$, then show that $n > 1$ is either a prime or a product of two primes.

(ii) The number $\sqrt{p}$ is irrational for any prime p.

(iii) For $n \geq 2$, $\sqrt[n]{n}$ is irrational.

(iv) Any three-digit composite number must have a prime factor less than or equal to 31.

(v) For $n > 2$, there exists a prime p satisfying $n < p < n!$.

(vi) For $n > 1$, every prime divisor of $n! + 1$ is an odd integer greater than n.

(vii) Let p_n denote the n^{th} prime number.

  (a) $p_n > 2n - 1$, for $n \geq 5$.

  (b) For any $n \geq 1$, number $p_1 p_2 \dots p_n + 1$ is never a perfect square.

  (c) For any $n \geq 1$, the number

$$\frac{1}{p_1} + \dots + \frac{1}{p_n}$$

  is never an integer.

3 Goldbach conjecture

1. Establish the following assertions.

   (i) The sequence

$$(n+1)! - 2, (n+1)! - 3, \dots, (n+1)! - (n+1)$$

   produces n consecutive composite integers for $n > 2$.

   (ii) If p_n denotes the n^{th} prime number, then for $n \geq 3$,

$$p_{n+3}^2 < p_n p_{n+1} p_{n+2}.$$

(iii) If p is a prime and $p \nmid b$, then every p^{th} term of the arithmetic progression

$$a, a+b, a+2b, \ldots$$

is divisible by p.

(iv) The number 13 is the largest prime that can divide two successive integers of the form $n^2 + 3$.

(v) For any integer $k > 0$, the arithmetic progression

$$a+b, a+2b, a+3b, \ldots,$$

where $\gcd(a, b) = 1$, has k consecutive terms that are composite.

(vi) There are infinitely many primes ending in 33. [Hint: Apply Dirichlet's theorem].

(vii) There exists infinitely many primes that contain, but do not end in the block of digits 123456789.

2. The *Bertrand conjecture* states that there exists at least one prime in the interval $(n, 2n)$, for $n \neq 2$. Assuming that the Bertrand conjecture holds true, establish the following assertions.

   (a) Let p_n denotes the n^{th} prime number. Then $p_n < 2^n$, for $n \geq 2$.

   (b) For $n > 3$,

   $$p_n < \sum_{i=1}^{n-1} p_i.$$

   (c) For every $n \geq 2$, there exists a prime p with $p \leq n < 2p$.